Dual space: — let L^* be the set of all linear functionals on a linear space $L(F)$. If $f_1, f_2 \in L^*$
i.e. f_1, f_2 are linear functionals on L,
let us define the mapping f_1+f_2 by
$$(f_1+f_2)(x) = f_1(x) + f_2(x) \ \forall x \in L.$$
Also if $\alpha \in F$ and $f \in L^*$, let us define the mapping αf by $(\alpha f)(x) = \alpha f(x) \ \forall x \in L$.
both f_1+f_2 and αf are linear functionals on L.
Also for these linear operations L^* is a linear space over the field F. The linear space L^* is called the dual space of L. It is also called the algebraic dual or the algebraic conjugate of L.

Theorem (A): — let $L(F)$ be an n-dimensional linear space and let $B = \{x_1, \ldots, x_n\}$ be an ordered basis for L. If $\{\gamma_1, \ldots, \gamma_n\}$ is any ordered set of n scalars, then there exists a unique linear functional on L such that
$$f(x_i) = \gamma_i, \quad i = 1, \ldots, n.$$

Proof: — Existence of f. let $x \in L$. Since B is a basis for L, therefore there exists unique scalars $\alpha_1, \ldots, \alpha_n$ such that $x = \alpha_1 x_1 + \ldots + \alpha_n x_n$.
For this vector x, let us define $f(x) = \alpha_1\gamma_1 + \ldots + \alpha_n\gamma_n$.
obviously $f(x)$ as defined by us is a unique element of F. Therefore f is a mapping from L into F. The unique representation of $x_i \in L$ as a linear combination of

Vectors belonging to the basis B is $x_i = 0x_1 + 0x_2 + \cdots + 1x_i + 0x_{i+1} + \cdots + 0x_n$.

Therefore ~~according to our d~~ according to definition of f, we have

$$f(x_i) = 0y_1 + 0y_2 + \cdots + 1y_i + 0y_{i+1} + \cdots + 0y_n$$

i.e. $f(x_i) = y_i$, $i = 1, \cdots, n$

Now we shall show that f is a linear functional.

Let $\alpha, \beta \in F$ and $x, y \in L$, Let

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n, \quad y = \beta_1 x_1 + \cdots + \beta_n x_n.$$

Then

$$f(\alpha x + \beta y) = f[\alpha(\alpha_1 x_1 + \cdots + \alpha_n x_n) + \beta(\beta_1 x_1 + \cdots + \beta_n x_n)]$$

$$= f[(\alpha\alpha_1 + \beta\beta_1)x_1 + \cdots + (\alpha\alpha_n + \beta\beta_n)x_n]$$

$$= (\alpha\alpha_1 + \beta\beta_1)y_1 + \cdots + (\alpha\alpha_n + \beta\beta_n)y_n$$

$$= \alpha(\alpha_1 y_1 + \cdots + \alpha_n y_n) + \beta(\beta_1 y_1 + \cdots + \beta_n y_n)$$

$$= \alpha f(x) + \beta f(y).$$

Therefore f is a linear functional on L. Thus there exists a linear functional f on L such that

$$f(x_i) = y_i, \quad i = 1, \cdots, n.$$

Uniqueness of f: Let g be a linear functional on L such that $g(x_i) = y_i$, $i = 1, \cdots, n$. For any vector

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n \in L,$$

we have,

$$g(x) = g(\alpha_1 x_1 + \cdots + \alpha_n x_n)$$

$$= \alpha_1 g(x_1) + \cdots + \alpha_n g(x_n) \quad [\because g \text{ is linear}]$$

$$= \alpha_1 y_1 + \cdots + \alpha_n y_n \quad [\text{by def of } g]$$

$$= f(x) \quad (\text{by def. of } f)$$

Thus $g(x) = f(x) \ \forall\, x \in L$. Therefore $g = f$ and f is unique.

Theorem (B) :- Let $B=\{x_1, —, x_n\}$ be a basis for an n- dimensional linear space $L(F)$. Then there is a uniquely determined basis $B^*=\{f_1, — f_n\}$ for L^* such that $f_i(x_j)=\delta_{ij}$. Consequently the algebraic dual of an n- dimensional space is n-dimensional.

The basis B^* is called the dual basis of B.

Proof :- $B=\{x_1, —, x_n\}$ is an ordered basis for L. Therefore by above theorem, there exists a unique linear functional f_1 on L such that $f_1(x_1)=1, f_1(x_2)=0, —, f_1(x_n)=0$ where $\{1, 0, —, 0\}$ is an ordered set of n scalars. In fact, for each $i=1, —, n$ there exists a unique linear functional f_i on L such that

$$f_i(x_j)=\begin{cases}0 & \text{if } i\neq j\\ 1 & \text{if } i=j\end{cases}$$

$$\text{i.e. } f_i(x_j)=\delta_{ij} \quad \text{———} \quad (1)$$

Where $\delta_{ij}\in F$ is kronecker delta i.e. $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$.

Let $B^*=\{f_1, —, f_n\}$. Then B^* is a subset of L^* containing n elements of L^*. We ~~show~~ shall show that B^* is a basis for L^*.

First we shall show that B^* is linearly independent.

Let $\gamma_1 f_1 + ——— + \gamma_n f_n = \hat{0}$ (zero vector of L^*)

$\Rightarrow (\gamma_1 f_1 + ——— + \gamma_n f_n)(x) = \hat{0}(x) \ \forall x\in L$

$\Rightarrow \gamma_1 f_1(x) + ——— + \gamma_n f_n(x) = 0 \ \forall x\in L$

$[\because \hat{0}(x)=0]$

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Expt. No. / Name : $\Rightarrow \sum_{i=1}^{n} \gamma_i f_i(x) = 0 \ \forall\, x \in L$

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$\Rightarrow \sum_{i=1}^{n} \gamma_i f_i(x_j) = 0, \ j = 1 \text{ --- } n$ (Putting $x = x_j$)

$\Rightarrow \sum_{i=1}^{n} \gamma_i g_{ij} = 0, \ j = 1 \text{ --- } n. \Rightarrow c_j = 0, \ j = 1, \text{ --- } n.$

$\Rightarrow f_1, \text{ --- }, f_n$ are linearly independent.

Now we shall show that B^* generates L^*. Let f be any element of L^*. The linear functional f will be completely determined if we define it on a basis for L. So let

$$f(x_i) = \alpha_i, \ i = 1, \text{ --- } n \quad \text{--- (2)}$$

We shall show that

$$f = \alpha_1 f_1 + \text{ --- } + \alpha_n f_n = \sum_{i=1}^{n} \alpha_i f_i.$$

We know that the two linear functionals on L are equal if they agree on a basis for L. So let $x_j \in B$ where $j = 1, \text{ --- }, n$. Then

$$\left(\sum_{i=1}^{n} \alpha_i f_i\right)(x_j) = \sum_{i=1}^{n} \alpha_i f_i(x_j) = \sum_{i=1} \alpha_i g_{ij} = \alpha_j.$$

On summing with respect to i and remembering that $g_{ij} = 1$ when $i = j$ and $g_{ij} = 0$ when $i \neq j$. But $\alpha_j = f(x_j)$ from (2). Thus $(\sum \alpha_i f_i)(x_j) = f(x_j) \ \forall\, x_j \in B$. Therefore $f = \sum \alpha_i f_i$. Thus every element f in L^* can be expressed as a linear combination of $f_1, \text{ --- }, f_n$. Therefore B^* spans L^*. Hence B^* is a basis for L^*.

Now $\dim L^*$ = number of distinct elements in $B^* = n$.

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Teacher's Signature